1 Retirement Models

We can begin enriching the life-cycle model by adding more realistic features. In particular, we focus on modeling retirement and public policy programs, such as investment retirement accounts (IRA's) and Social Security. How households prepare for the phase of life, or their financial risk exposure during retirement years has important public policy implications.

As we showed in the presentation of the life-cycle model, productivity tends to increase early in life, peak out, then decline in the later years. In reality, individuals tend to explicitly exit the labor market around age 65 and enter into a period of life that we call *retirement.* In this section, formalize this concept of retirement and introduce several concepts relating to public policy.

Consider the life-cycle model introduced in the previous section:

$$V_t(a_t) = \max_{n_t, a_{t+1}} \quad u(c_t, n_t) + s_{t+1}\beta V_{t+1}(a_{t+1}) \tag{1}$$

$$\text{s.t.} \;\; c_t = wz_t n_t + Ra_t - a_{t+1} \tag{2}$$

$$0 \leq n_t \leq 1, \tag{3}$$

$$\text{and} \;\; a_{T+1} \geq 0. \tag{4}$$

Notice here that the utility function has been generalized to account explicitly for labor supply. This does not really matter until we specify the utility function, since we can always choose a functional form that does not account for labor supply. Nevertheless, it helps to begin considering the problem more generally.

In this section, we will account for retirement by assuming that agents retire exogenously at age T_r. In the working years leading up to age T_r, agents' productivity z_t will take positive values. However, we will assume that $z_t = 0$ every period beginning in T_r.

1.1 Social Security

The Social Security program in the United States is a multifaceted social insurance program that depends in many ways on households' decisions. That said, we can begin studying the Social Security program in the context of our life-cycle model by considering the two simple features of the program from a public finance perspective. First, the government pays households some amount of income once they reach a certain age. Second, this expenditure is financed through a Social Security tax on households' labor income.

We begin by assuming that households receive a lump-sum Social Security benefit ss in retirement. Social Security outlays are financed by a proportional labor income tax τ_{ss}. We will want to begin thinking of the life-cycle as two discrete phases of life, but we want to continue generalizing the framework in a unified way. Thinking about the problem in a generalized way will help us implement efficient code-writing when we compute model. Specifically, we can write the generalized problem now by simply modifying the budget constraint (2) as follows:

$$c_t = (1 - \tau_{ss})wz_t n_t + Ra_t - a_{t+1} + \mathbb{1}_{t \geq T_r} ss, \tag{5}$$

where $\mathbb{1}$ is an indicator function that takes a value of one if the subscripted condition is true, and zero otherwise. Since we assumed that labor productivity is zero starting at age T_r, the budget constraint is correctly specified at every age.

While accounting for Social Security in the agent's problem is a simple modification of the individual's budget constraint, the feasibility of a balanced government budget relies heavily on demographics. In particular, suppose that μ_t denotes the population weight corresponding to

Equation 11 in the previous section. Recall that these population weights relied on stationarity of the age distribution, which we continue assuming here. Then, we can write the government's budget constraint as:

$$\sum_{t=1}^{T_r-1} \mu_t \tau_{ss} w z_t n_t = \sum_{t=T_r}^{T} \mu_t ss, \tag{6}$$

where the left-hand side of the equation shows government revenue, and the right-hand side of the equation shows government outlays.

We can consider different parts of (6) to get two important statistics of interest. First, we define the *replacement ratio* as:

$$\textit{Replacement Ratio} = \frac{\sum_{t=T_r}^{T} \mu_t ss}{\sum_{t=1}^{T_r-1} \mu_t w z_t n_t}, \tag{7}$$

which gives the average percentage of labor income that is replaced by Social Security income during retirement years. This is usually interpreted as a measure of the generosity of the Social Security system. Second, we have the dependency ratio:

$$\textit{Dependency Ratio} = \frac{\sum_{t=T_r}^{T} \mu_t}{\sum_{t=1}^{T_r-1} \mu_t}, \tag{8}$$

which simply gives the percentage of individuals who depend on the rest of the population to finance their benefit. Often times, the dependency ratio is provided as a demographic measure of the resources available to fund the program. A decline in the population growth rate or an increase in life expectancy will drive up the dependency ratio. Also notice that the dependency ratio is a function of the retirement age T_r. If the government chooses to increase the Social Security retirement age, *ceteris paribus*, the dependency ratio would decline.

1.2 Retirement Contributions and Investment Retirement Accounts

To encourage individuals to save more for retirement, the U.S. tax code favorably treats retirement account contributions during working years and subsequent withdraws in retirement years. A 401(k) is type of retirement account that allows for favorable tax treatment towards retirement contributions from labor income. By contrast, a Roth or Traditional Investment Retirement Account (IRA) is a type of retirement account that does not tax your investment return upon retirement. Each of these accounts have limitations and idiosyncrasies, but we can model the general aspects of the programs.

1.2.1 Retirement Contributions

Instead of thinking about separate retirement and non-retirement savings accounts, we will simply think about savings as contributions to a retirement account that can only be accessed in retirement years. That means that agents can only consume out of their labor income during working years. Let ϕ represent the percentage of labor income that the agent chooses to allocate towards the retirement account. Also, since we will need to evaluate the policy in the context of a tax deduction, we introduce an arbitrary income tax function $\tau(\cdot)$, which we will specify later. We can rewrite the problem of the working-age agent as:

$$V_t(a_t^R) = \max_{n_t, \phi_t} \; u(c_t, n_t) + s_{t+1} \beta V_{t+1}(a_{t+1}^R) \tag{9}$$

$$\text{s.t. } c_t = wz_tn_t(1-\phi) - \tau(wz_tn_t(1-\phi)), \text{ (budget constraint)} \tag{10}$$

$$a^R_{t+1} = Ra^R_t + wz_tn_t\phi, \text{ (law of motion)} \tag{11}$$

$$\text{and } 0 \leq n_t \leq 1, \text{ (time allocation)} \tag{12}$$

where we rename the savings variable a^R to distinguish between retirement savings and more general savings that we have considered up to this point. Notice that even though the agent can not access the retirement account during working years, retirement savings remains as a state-space variable. Moreover, we can be confident that the value function for an age t individual will be increasing in retirement saving a^R_t.

The budget constraint (10) shows how saving fraction ϕ of income not only reduces the amount of resources available for consumption today, but it also reduces taxable income. This tax deduction is the channel through which policymakers create incentives for individuals to save more for retirement. Equation (11) shows how the retirement account grows over time. In particular, at any point in time, the retirement account grows according to the gross return on assets plus any contributions.

The current framework allows us to evaluate government policies, such elimination of employee contributions. It can also tells us how retirement savings might be influenced by changes in the tax rate. The U.S. tax code generally caps contributions to these accounts to prevent tax arbitrage opportunities that create loopholes in the tax code. Accordingly, if we wanted to understand the consequences of limiting retirement contributions, we could modify the constraints accordingly.

Now that we have considered the working-age individual's problem, we can complete the life-cycle problem by considering the retiree's problem:

$$V_t(a^R_t) = \max_{a^R_{t+1}} \; u(c_t, 0) + s_{t+1}\beta V_{t+1}(a^R_{t+1}) \tag{13}$$

$$\text{s.t. } c_t = Ra^R_t - a^R_{t+1} - \tau(Ra^R_t - a^R_{t+1}) \text{ (budget constraint)} \tag{14}$$

$$\text{and } a^R_{T+1} \geq 0. \text{ (no-Ponzi condition)} \tag{15}$$

This problem now resembles our original life-cycle model with a few important exceptions. First, notice that even though our choice variable changed from labor supply and contribution percentage in the working-age problem, now the choice is purely savings. Next, since retirees in this model do not work, we have simply replaced labor supply with zero in the utility function. Finally, notice that we now tax retirement income - both the principal and interest. This is an important distinguishing feature between these types of accounts and IRA accounts.

1.2.2 Investment Retirement Accounts

In many ways, IRA's are the opposite of retirement contributions. Whereas retirement contributions are tax deductible and later taxed during retirement years, IRA contributions are not tax deductible but can be withdrawn free of taxation during retirement. If we wanted the model to account for this, we would modify the working-age problem as follows:

$$V_t(a^R_t) = \max_{n_t, q_t} \; u(c_t, n_t) + s_{t+1}\beta V_{t+1}(a^R_{t+1}) \tag{16}$$

$$\text{s.t. } c_t = wz_tn_t - \tau(wz_tn_t) - q_t, \text{ (budget constraint)} \tag{17}$$

$$a^R_{t+1} = Ra^R_t + q_t, \text{ (law of motion)} \tag{18}$$

$$\text{and } \ 0 \leq n_t \leq 1, \ \text{(time allocation)} \tag{19}$$

where q_t is the new choice variable that denotes the magnitude of the contribution to the IRA. Suppose we wanted to account for the maximum contributions imposed by the government. All that we would have to do is add the constraint $q_t \leq \bar{q}$, where $\bar{q}$ sets the maximum annual contribution. The corresponding retiree's problem can be expressed as:

$$V_t(a_t^R) = \max_{a_{t+1}^R} \ u(c_t, 0) + s_{t+1}\beta V_{t+1}(a_{t+1}^R) \tag{20}$$

$$\text{s.t. } \ c_t = Ra_t^R - a_{t+1}^R \ \text{(budget constraint)} \tag{21}$$

$$\text{and } \ a_{T+1}^R \geq 0. \ \text{(no-Ponzi condition)} \tag{22}$$