1 Production

This section begins a discussion on firms and production in dynamic macroeconomic models. We start by considering the simple firms that are commonly assumed in dynamic general equilibrium models. Accordingly, we conclude the development of our models by showing how prices clear... Next, we return to partial equilibrium to consider more complex firm models. Abstracting from general equilibrium allows us to focus on the interesting features of firm models.

1.1 General Equilibrium

Up to this point, we have assumed that wages and interest rates are exogenous. In a general equilibrium framework, households supply labor and rent their capital to firms, while firms hire labor and rent household capital to produce output. We introduce the aggregate production function:

$$Y = F(K, L), \tag{1}$$

where K is capital, L is efficient labor, Y is output, and F is the production function. In most general equilibrium models, we will assume that the firm's production function takes the Cobb-Douglas form, $F(K, L) = K^\alpha L^{1-\alpha}$. If firms take wages w, interest rates R, and the capital depreciation rate δ as given, the profit maximization problem can be stated as follows:

$$\max_{K,L} K^\alpha L^{1-\alpha} - RK - wL + (1-\delta)K, \tag{2}$$

where the last term is the undepreciated capital that is returned to households at the end of the period. The first order conditions of the firm are:

$$R = \alpha K^{\alpha-1} L^{1-\alpha} + 1 - \delta \tag{3}$$

$$w = (1-\alpha) K^\alpha L^{-\alpha}. \tag{4}$$

Notice that each of these first-order conditions can be written as a function of the capital-to-labor ratio:

$$R = \alpha \left(\frac{K}{L}\right)^{\alpha-1} + 1 - \delta \tag{5}$$

$$w = (1-\alpha) \left(\frac{K}{L}\right)^{\alpha}. \tag{6}$$

This means that the firm's optimal solution reduces to choosing the right capital-to-labor ratio.

Now think back to household optimization problem. We can consider a life-cycle framework or an infinite horizon - it doesn't really matter. All that matters is that households take interest rates and wages as given and choose optimally. Let x generically denote the household's state space, and let $\mu(x)$ denote the measure of individuals with state x. Denote aggregate household savings as $K = \sum_{x \in \mathcal{X}} \mu(x) a'(x)$ and aggregate efficient labor as $L = \sum_{x \in \mathcal{X}} \mu(x) z n(x)$. Notice that we have used the variable K to denote aggregate savings. We model the economy by assuming that individuals save by renting their assets to firms.

Suppose that we solved for aggregate savings and aggregate labor and took the ratio: $\frac{K}{L}$. We could substitute this ratio into the first order conditions of the firm (5)-(6). If the interest rates and wages implied by this capital-to-labor ratio were the same as the interest rates and wages used to solve household optimization problem, then we would have a *competitive equilibrium* - a solution to the general equilibrium problem. How can we find the equilibrium interest rate and wage otherwise? The following algorithm usually leads to a solution:

1. Guess a capital-to-labor ratio: $\frac{K_0}{L_0}$ and solve for the corresponding interest rate and wage using (5)-(6).

2. Use this interest rate $R(\frac{K_0}{L_0})$ and wage $w(\frac{K_0}{L_0})$ to solve the household optimization problem. Solve for aggregates and take the ratio: $\frac{\tilde{K}_0}{\tilde{L}_0}$.

3. If $|\frac{K_0}{L_0} - \frac{\tilde{K}_0}{\tilde{L}_0}| < \varepsilon$, then an equilibrium has been successfully approximated. Otherwise, let the next guess of the capital-to-labor ratio be a weighted average of these two values: $\frac{K_1}{L_1} = \rho\frac{K_0}{L_0} + (1-\rho)\frac{\tilde{K}_0}{\tilde{L}_0}$ and start back at step 1.

Note that this process clears two markets - the labor market and the capital market. However, this model economy has three markets, labor, capital, and goods. The algorithm proposed above chooses over wages and interest rates to clear the labor market and capital market simultaneously. We do not have a price for the consumption good because every price in the economy is relative to the price of consumption, which is normalized to one. Further, by Walras' Law, clearing $N-1$ markets ensures that N markets clear. In our case, we have three markets, so clearing labor and capital markets ensures that the goods market clears as well.

1.2 Heterogeneous Firms

Firms in the previous section were useful for solving a general equilibrium model, but they did not capture many interesting features of firm operation. In fact, whereas the firms in the previous section solved a static optimization problem, we will now consider dynamic investment decisions of firms. To that extent, we will now assume that firms own their capital.

The goal of firms in a static setting is to maximize profit. In reality, these profits accumulated by firms are paid to the firms' owners. Since firms in the previous section only operated for one period, there was no need to solve a dynamic optimization problem. However, many of the key decisions that firms make, especially with regards to investment, require an intertemporal trade-off which requires the formulation of a Bellman's equation. In particular, we will assume that firms take the interest rate $1+r$ as given and maximize shareholder value by maximizing the expected discounted present value of all issued dividends:

$$E\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t d_t\right]. \tag{7}$$

We begin by considering a production Cobb-Douglas production function as in the previous section:

$$y = zk^{\alpha_1}n^{\alpha_2}, \tag{8}$$

where z is a total factor productivity shock, $\alpha_1 + \alpha_2 \leq 1$, and lower-case variables are used to denote individual, rather than aggregate values. Similar to the idiosyncratic shocks faced by households, firms' idiosyncratic shocks will imply a corresponding distribution over firm variables. We will assume that capital that is chosen in the current period goes into production in the following period. Further, changing the capital stock incurs capital adjustment costs $C(k,k')$. Letting the firm's discount factor be $\frac{1}{1+r}$, where r is the net interest rate, we can write the firm's Bellman equation as:

$$V(z,k) = \max_{k',n} zk^{\alpha_1}n^{\alpha_2} - wn - C(k,k') - (k' - (1-\delta)k) + \frac{1}{1+r}E\left[V(z',k')\right]. \tag{9}$$

The term $(k' - (1-\delta)k)$ in (9) represents firm investment and explains how firms accumulate capital. Notice that the labor demand decision in (9) is a static optimization problem. The derivative of the continuation value with respect to labor choice is zero. Accordingly, optimal labor demand is:

$$n^* = \left[\frac{w}{\alpha_2 z k^{\alpha_1}}\right]^{\frac{1}{\alpha_2 - 1}}. \tag{10}$$

To simplify notation, we substitute (10) into the contemporaneous payoff function and introduce the indirect profit function:

$$\pi(z,k) = z k^{\alpha_1} n^{*\alpha_2} - w n^*, \tag{11}$$

where n^* is determined in (10). We can now write the Bellman equation as a single-dimensional optimization problem:

$$V(z,k) = \max_{k'} \pi(z,k) - C(k,k') - (k' - (1-\delta)k) + \frac{1}{1+r} E\left[V(z',k')\right]. \tag{12}$$

Thinking back to (7), we see from (15) that the dividend (or equivalently, equity payment) is $d = \pi(z,k) - C(k,k') - (k' - (1-\delta)k)$. If this value is negative, it is assumed that the firm has devalued the contemporaneous equity payment by issuing new equity shares to maximize firm value. Such a method of financing investment is called *equity financing.*

1.2.1 Tax Policy

The framework we developed allows us to consider several public policy issues with regards to corporate taxation. Specifically, we will consider a corporate income tax function $\tau_c(\cdot)$, which is a tax on corporate profits. The government allows the deduction of several items from taxable income, each having their own behavior distortions. For example, investment expensing or accelerated depreciation each distort the investment decision. Deductibility of interest payment on debt, by contrast, encourages firms to choose debt financing, rather than equity financing. The Modigliani-Miller theorem of corporate finance suggests that firms should generally be indifferent between debt and equity financing. However, debt interest deductibility breaks down this result. We will think more about debt in a later section, and focus on corporate income taxes and investment expensing here.

Now that we have defined a corporate income tax function $\tau_c(\cdot)$, we must consider what is defined as corporate income. Inherent in the term $\pi(z,k)$ are total revenue and the deduction of the wage bill, but we will also assume that any capital adjustment costs can be deducted. Following Hennessy and Whited, we will assume that capital depreciation δk is also deductible. We proceed with the assumption that actual capital depreciation equals accounting depreciation. In reality, these two things can differ, and the government generally allows for accelerated depreciation. Finally, we will assume that some portion of investment can be deducted from the corporate tax bill. This is an important component of the current proposed legislative tax reform bill. Like we said earlier, investment is the component of the firm's contemporaneous payoff equal to $k' - (1-\delta)k$. Accordingly, we will assume that some portion ϕ of *new* investment can be expensed, so that the new investment deduction is $\phi \max\left\{(k' - (1-\delta)k), 0\right\}$. Finally, we can define corporate taxable income as:

$$y(z,k,k') = \pi(z,k) - C(k,k') - \delta k - \phi \max\left\{(k' - (1-\delta)k), 0\right\}. \tag{13}$$

Then, we can write the firm's tax bill as

$$\tau_c(y(z,k,k')) \tag{14}$$

and generalize the firm's Bellman equation as:

$$V(z,k) = \max_{k'} \pi(z,k) - C(k,k') - (k' - (1-\delta)k) - \tau_c(y(z,k,k')) + \frac{1}{1+r} E\left[V(z',k')\right]. \tag{15}$$

1.2.2 Variable Interpretation

Let's return back down to Earth for a moment. Why are we developing these models? Models are only useful to the extent that they help us understand the real world. By understanding how these model variables relate to real-world accounting values, we ensure that this mathematical approach is more than just a hazing ritual in the economics profession. To bridge the gap between quantified theoretical models and real-world accounting variables, we consider a few of the variables from the table on p. 1152 of Hennessy and Whited (2005)[1]:

Investment to the book value of assets	$(k' - (1-\delta)k)/k$
Tobin's Q	$V(z,k)/k$
EBITDA to the book value of assets	$\pi(z,k)/k$
Equity issuance to the book value of assets	$\left[\pi(z,k) - C(k,k') - (k' - (1-\delta)k) - \tau_c(y(z,k,k'))\right]/k$

Each of the above values are ratios that we could construct from a company's 10-K form. In practice, the parameter values of this model are chosen so that the model-generated statistics are sufficiently close to the values observed in the real world. A systematic execution of such an exercise is known as simulated method of moments estimation. This method involves choosing the parameters of the model to minimize the weighted disparity between model moments and corresponding data moments. A more casual approach is known as calibration, which involves manually picking parameters or choosing them from the literature and comparing model-generated statistics to corresponding data statistics for model validation.

1.2.3 Equity Financing

We mentioned earlier that new equity issuance was implied by a negative dividend, so we now explicitly account for the associated issuance costs incurred by firms. In corporate finance, new equity issuance is known as underwriting syndication, and the associated costs can be incurred for a number of reasons. These costs can either be directly incurred by the firm or implicitly through undervalued equity offered through syndication. Regardless, we assume that new equity issuance is costly and proportional to the magnitude of the equity issued.

We will find it useful to begin denoting concurrent dividends or equity payments as follows:

$$e(z,k,k') = \pi(z,k) - C(k,k') - (k' - (1-\delta)k) - \tau_c(y(z,k,k')). \tag{16}$$

This notation allows us to introduce a new term corresponding to equity issuance costs. In

[1] Hennessy, Christopher A., and Toni M. Whited. "Debt dynamics." *The Journal of Finance* 60, no. 3 (2005): 1129-1165.

particular, we introduce equity issuance cost function $\lambda(\cdot)$:

$$\lambda(e(z,k,k')) = \begin{cases} -|a_0 + a_1 e(z,k,k') + a_2 e(z,k,k')^2|, & \text{if } e(z,k,k') < 0 \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

Notice that we have imposed the assumption that equity issuance costs can be represented by a second order polynomial. Up to this point, we have avoided discontinuities in the objective function to help ensure that local optimal solutions are also globally optimal. If $a_0 \neq 0$, however, we have a discontinuity at the origin, which creates an issue for our computational optimization routine. One simple solution is to restrict $a_0 = 0$ to ensure that the objective function is continuous at the origin, but that assumption may be unrealistic. Instead, it might make sense to compute optimization problem twice, initializing it from each side of the origin. For now, we can just think of equity issuance costs more generally as some costs associated with issuing negative equity and write our new, compact Bellman equation, which includes the equity issuance costs as follows:

$$V(z,k) = \max_{k'} e(z,k,k') + \lambda(e(z,k,k')) + \frac{1}{1+r} E\left[V(z',k')\right]. \tag{18}$$

1.2.4 Debt Accumulation

As of now, our model includes two different ways for firms to finance investment. If the firm issues a positive dividend, we interpret any investment expenditures as being financed by concurrent revenue. However, if the firm issues a negative dividend, we assume that the firm has issued new shares of equity to finance investment. In this section, we introduce a third channel available to firms for financing their expenditures - debt accumulation.

In the absence of any government intervention, the Modigliani-Miller theorem suggests that firms should be indifferent between debt and equity financing. In the United States, however, the government allows firms to deduct debt interest payments from taxable income, motivating firms to favor debt accumulation. Following Hennessy and Whited (2005), and introduce debt b_t as a new dimension of the firm's state space. In this presentation of debt, however, we will explicitly consider debt interest payment as its own independent component of the tax function as follows:

$$e(z,k,k',b,b') = \pi(z,k) - C(k,k') - (k' - (1-\delta)k) - (1+r)b + b' - \tau_c(y(z,k,k')), \tag{19}$$

where taxable income is now:

$$y(z,k,k',b) = \pi(z,k) - C(k,k') - \delta k - \phi \max\left\{(k' - (1-\delta)k), 0\right\} - \max\left\{rb, 0\right\}. \tag{20}$$

In the same spirit as the no-Ponzi scheme that we introduced in the life-cycle models, we must restrict the amount of debt that a firm can issue. We want to choose a value such that a firm could always repay the debt in a worst case scenario. We will say that the worst case scenario is liquidation, and this constraint represents the firm's collateral. Following Hennessy and Whited (2005) and DeAngelo, DeAngelo, and Whited (2011), we pick a value such that a firm forfeits its cash flow and undepreciated capital, which must be sold in a *firesale* at a fraction s of its value to the firm:

$$b' \leq s(1-\delta)k' + \pi(\underline{z},k') - \tau_c(y(\underline{z},k',0,b')), \tag{21}$$

where $\underline{z}$ represents the worst possible productivity shock that a firm could incur. Notice that the collateral constraint itself depends on the amount of debt and capital chosen. In other words, debt incurred just to pay a higher dividend would face a tighter constraint than debt issued to finance capital investment. We can now write the Bellman equation using our concise as:

$$V(z,k,b) = \max_{k',b'} \left\{ e(z,k,k',b,b') + \lambda(e(z,k,k',b,b')) + \frac{1}{1+r} E\left[V(z',k',b')\right] \right\} \tag{22}$$

$$\text{s.t. } (21) \tag{23}$$

1.2.5 Risky Debt

While the collateral constraint in the previous section ensures that a firm could always pay back its debt, it may not accurately represent situations in which firms become financially distressed. In fact, if firms incur too much debt and subsequently experience a sharp decline in demand for their product, then the firm value may decline to zero. In such a situation, a firm may choose to default on its debt, in which case the creditor may end up with the remaining assets. We formalize this concept now.

Because of limited liability, the most that a firm's shareholder can lose is the value of the firm. This means that the firm's value is bounded below by zero, and in such cases a firm may choose to default. Now suppose that the creditor can anticipate outcomes in which the firm would choose to default, and calculate the probability that a firm will default, conditional on the amount of debt that it incurs. Knowing that the it will acquire the firm's assets, the creditor charges an interest rate corresponding to the likelihood of default: $r(z,k',b')$. The creditor conditions the interest rate on the current productivity shock z because it is the best indicator of the future productivity shock by virtue of the Markov process. The interest rate also depends on the future capital stock because it determines the collateral acquired in the event of liquidation. Finally, the interest rate depends on the amount of debt incurred because it helps the creditor predict the likelihood of default.

Because the firm does not pay the interest on debt incurred until the next period, it is useful to add future interest rate as another state space dimension. Then, we can solve for next period's interest rate in the current period, and simply interpolate over the continuation value by solving for the appropriate interest rate determined by the creditor. At this point, we have all of the elements that we need to write the firm's Bellman equation, accounting for endogenous default as follows:

$$V(z,k,b,r) = \max\left\{0, \max_{k',b'} e(z,k,k',b,b',r) + \lambda(e(z,k,k',b,b',r)) + \frac{1}{1+r} E\left[V(z',k',b',r')\right]\right\} \tag{24}$$

$$\text{s.t. } r' = r(z,k',b'), \tag{25}$$

where $r(z,k',b')$ is determined by the creditor.

To solve for this risky interest rate, suppose that in default states, a creditor would obtain the firms assets and sell them off at a discount. Following Hennessy and Whited (2007), let $R(z,k) = s(1-\delta)k + \pi(\underline{z},k) - \tau_c(y(\underline{z},k',0,b'))$, where the term $R(z,k)$ just denotes the value of the collateral in (21) obtained by the creditor in the event of default. Since the creditor knows ahead of time the states $\{z',k',b'\}$ in which the firm would choose to default, it can calculate the probability of default, conditional on the current shock z. Let $S_{solvency}$ denote the portion of the state space where firm optimally chooses not to default, and let $S_{default}$ denote the portion of the state space where the firm optimally chooses to default, where the entire state space $S = S_{solvency} \cup S_{default}$. Then the firm should choose an interest rate corresponding to risky debt such that it would be indifferent between investing in a risk-free asset earning r and the risky asset. Following Strebulaev and Whited (2012) indifference can be represented by the equation:

$$(1+r)b' = \int_{s' \in S_{solvency}} (1+\tilde{r})b' dF(z'|z) + \int_{s' \in S_{default}} R(z', k') dF(z'|z) \qquad (26)$$

where $r(z, k', b')$ is the value $\tilde{r}$ that solves (26).