1 Life-cycle Model

In this section, we formalize the life-cycle utility maximization problem. Specifically, we will begin by assuming that the agent maximizes lifetime utility, which can be represented by the sum of discounted per-period utility as follows:

$$\max_{\{a_{t+1}\}_1^T} \sum_{t=1}^{T} \beta^{t-1} u(c_t) \tag{1}$$

$$\text{s.t. } c_t = wz_t n_t + Ra_t - a_{t+1}, \text{ (period t budget constraint)} \tag{2}$$

$$0 \leq n_t \leq 1, \text{ (time limitation)} \tag{3}$$

$$\text{and } a_{T+1} \geq 0, \text{ (no Ponzi scheme)} \tag{4}$$

where T is the maximum lifetime, β is the discount factor, and $u(\cdot)$ satisfies Inada conditions. The variables c_t, a_{t+1}, and n_t are consumption, savings, and labor supply chosen in period t, respectively. We assume that prices R and w are constant throughout the agent's lifetime. We introduce a new variable, z_t, which will represent life-cycle productivity. We can think of this as an increase in productivity and earning potential as age increases, followed by a decline towards retirement.

We note three important things here. First, we see that leisure is not in the utility function. Since utility is increasing in consumption, and consumption is increasing in labor, working more will always yield higher utility. Consequently, we know that optimal will labor supply will be the upper bound of the time endowment, i.e., $n_t^* = 1$ for all t. Second, we formalized a restriction on saving in the last period of life to ensure that individuals do not intend to bequeath any debt. Finally, since every dynamic problem needs initial conditions, a subtle parameter is initial wealth, a_1.

As we learned in the previous section, this problem can be represented by Bellman's equation in any arbitrary period t:

$$V_t(a_t) = \max_{a_{t+1}} \; u(c_t) + \beta V_{t+1}(a_{t+1}) \tag{5}$$

$$\text{s.t. constraints (2) - (4)}$$

Generally, when we write Bellman's equation, we assume that $V_{T+1} = 0$, so that the value function in the last period of life is

$$V_T(a_T) = \max_{a_{T+1}} \; u(c_T) = u(wz_T + Ra_T) \tag{6}$$

When solving this problem numerically, we have two options. We can either substitute (6) into the period T optimization problem, or we can simply assign $V_{T+1} = 0$ and allow the computer solve the optimization problem.[1]

2 Demographics in the Life-cycle Model

The previous section showed us how to solve a single individual's optimal lifetime utility assuming that the individual's survival probability was either inexplicit or somehow represented by the discount factor. Here, we will explicitly account for mortality in the agent's problem, and by doing so, we implicitly assume that individuals maximize lifetime *expected* utility. Then, we

[1] If you choose the latter approach, test to make sure the numerical solution, in fact, delivers $a^*_{T+1} = 0$.

will consider demographics from the perspective of aggregation. This basically means accounting for the distribution of individuals over age. To do this, we will need to consider both the mortality rate and the population growth rate.

We start by making a simplifying assumption that the population is normalized to one. For example, assume that individuals live for two periods and that two-thirds of individuals in the economy are age 1, while one-third are age 2. If age 1 and age 2 individuals have labor supply $n_1 = \frac{1}{2}$ and $n_2 = \frac{1}{4}$, respectively, then total labor in the economy is $N = \frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{4} = \frac{5}{12}$. Accounting for demographics like this will allow us to easily think about demographics as percentages of the total population.

We modify our Bellman equation by introducing a variable s_{t+1} accounting for the probability of surviving from period t to $t+1$ as follows:

$$V_t(a_t) = \max_{a_{t+1}} \quad u(c_t) + s_{t+1}\beta V_{t+1}(a_{t+1}) \tag{7}$$

$$\text{s.t. constraints (2) - (4)}$$

This ensures that the agent now properly accounts for survival probability. Next, we want to begin aggregating the agents' actions in a way that properly accounts for both survival probability and population growth. If the population growth rate were zero, and the survival probability were $s_{t+1} = 1$ for all t up to the maximum age T, then for any individual variable x_t, the aggregate variable would just be the weighted sum over variables of each age:

$$X = \sum_{t=1}^{T} \frac{1}{T} x_t. \tag{8}$$

We ensured that the population sums to one, as required by the normalization assumption, by weighing each cohort by $\frac{1}{T}$.

To derive population weights that account for survival, suppose that the measure of age 1 individuals (μ_1) is equal to 1: $\mu_1 = 1$. Then, if the probability of surviving to age 2 is s_2, it must be the case that the measure of age 2 individuals is $\mu_2 = \mu_1 s_2$. For example, suppose that the probability of surviving to age 2 is 99%. If we start of with a unit measure of individuals, then the age 2 individuals have population 0.99, and age 1 and age 2 individuals would have population 1.99. Generalizing this concept for any age t gives the population equation:

$$\mu_{t+1} = s_{t+1}\mu_t \tag{9}$$

Next, we want to account for the population growth rate. We will implicitly assume that the population is *stationary*, which just means that relative ages have converged to their long-run trend. The population grows at rate ν means that, in any period, age 1 population must be larger than age 2 population - even if the probability of surviving to age 2 is $\mu_2 = 1$. If $s_{t+1} = 1$, relationship between consecutive ages when the population grows is given by:

$$\mu_t = \mu_{t+1}(1 + \nu). \tag{10}$$

Then, accounting for survival, we can combine (9) and (10) to get:

$$\mu_{t+1} = \frac{s_{t+1}\mu_t}{1 + \nu}. \tag{11}$$

To compute these population weights, we only need the sequence of survival probabilities, $\{s_{t+1}\}_{t=1}^{T}$ and the population growth rate ν. We can start by assuming that $\mu_1 = 1$ and generating the sequence $\{\mu_2, \cdots, \mu_T\}$. To normalize the total population to 1, we just divide each weight by the sum of the weights. After doing this, each weight μ_t can be interpreted as the percentage of the population represented by each age group.

Once we have calculated the population weights $\{\mu_1, \cdots, \mu_T\}$, any individual actions x_t can be aggregated according to:

$$X = \sum_{t=1}^{T} \mu_t x_t. \tag{12}$$

In the life-cycle model presented in the previous section, we can calculate several aggregate quantities of interest: consumption, savings, labor hours, effective labor, and utility. Aggregate quantities in the economy corresponding to each of these values can be calculated as follows:

$$C = \sum_{t=1}^{T} \mu_t c_t^* \quad \text{(Aggregate Consumption)}$$

$$A = \sum_{t=1}^{T} \mu_t a_{t+1}^* \quad \text{(Aggregate Savings)}$$

$$N = \sum_{t=1}^{T} \mu_t n_t^* \quad \text{(Aggregate Labor Hours)}$$

$$Z = \sum_{t=1}^{T} \mu_t z_t n_t^* \quad \text{(Aggregate Effective Labor)}$$

$$W = \sum_{t=1}^{T} \mu_t V_t \quad \text{(Aggregate Welfare).}$$

Several things to notice here. First, since labor is supplied inelastically ($n_t^* = 1$ for all t), aggregate labor hours will be $N = 1$, which results from $\sum_{t=1}^{T} \mu_t = 1$. Next, we introduce the concept of aggregate effective labor. This is a measure of how efficient labor is being used for the purpose of production. We will have more to say about that later. Finally, notice that we introduced a mathematical concept of *aggregate welfare*. In our models, aggregate welfare will give us a measure of economic agents' well-being, which can be used to reach *normative* conclusions. Aggregate welfare provides a basis of comparison between alternative policy reforms and allows us to optimize over a variety of policy options.